Chapter 1

Rectangular Cartesian Co-ordinates

Contents

Rene' Descartes

Geometry is one of the most ancient branch of mathematics. A Systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician Rene' Descartes (1596-1650), in his book 'La Geometrie' which was published in 1637.

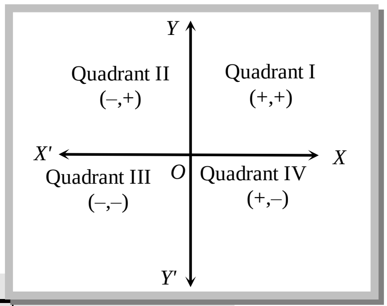
In order to relate algebra with geometry. Descartes established a relationship between the basic geometric concept of 'point' with basic algebraic entity 'number'. This relationship is called 'System of Co-ordinates'. Rene' Descartes related the position of a point with its distance from fixed line and its direction.

Leibnitz used the terms 'abscissa', ordinate and 'co-ordinate'. L' Hospital wrote (about 1700 A.D.) wrote an important text book on analytic geometry.

Rectangular Cartesian Co-ordinates

1.1 Introduction

Co-ordinates of a point are the real variables associated in an order to a point to describe its location in some space. Here the space is the two dimensional plane. The work of describing the position of a point in a plane by an ordered pair of real numbers can be done in different ways.



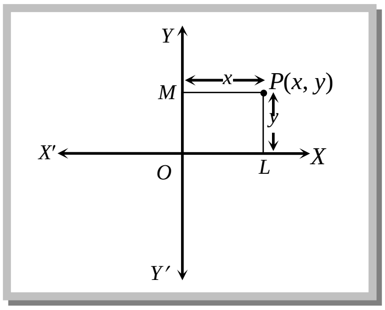
The two lines XOX' and YOY' divide the plane in four quadrants. XOY, YOX', $X'OY'$, $Y'OX$ are respectively called the first, the second, the third and the fourth quadrants. We assume the directions of OX, OY as positive while the directions of OX', OY' as negative.

Quadrant	*x*-coordinate	*y*-coordinate	point
First quadrant	+	+	(+,+)
Second quadrant	–	+	(–,+)
Third quadrant	–	–	(–,–)
Fourth quadrant	+	–	(+,–)

1.2 Cartesian Co-ordinates of a Point

This is the most popular co-ordinate system.

Let us consider two intersecting lines XOX' and YOY', which are perpendicular to each other. Let P be any point in the plane of lines. Draw the rectangle $OLPM$ with its adjacent sides OL,OM along the lines XOX', YOY' respectively. The position of the point P can be fixed in the plane provided the locations as well as the magnitudes of OL, OM are known.



Axis of *x* : The line XOX' is called axis of x.

Axis of *y* : The line YOY' is called axis of y.

Co-ordinate axes : x axis and y axis together are called axis of co-ordinates or axis of reference.

Origin : The point 'O' is called the origin of co-ordinates or the origin.

Oblique axes : If both the axes are not perpendicular then they are called as oblique axes.

Let $OL = x$ and $OM = y$ which are respectively called the abscissa (or x-coordinate) and the ordinate (or y-coordinate). The co-ordinate of P are (x, y).

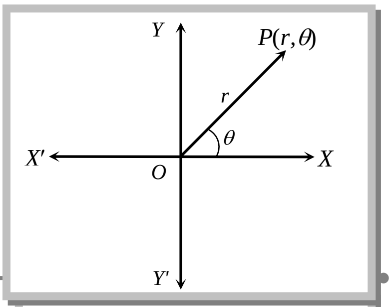
Note : ❑ Co-ordinates of the origin is (0, 0).

❑ The y co-ordinate of every point on x-axis is zero.

❑ The x co-ordinate of every point on y-axis is zero.

1.3 Polar Co-ordinates

Let OX be any fixed line which is usually called the initial line and O be a fixed point on it. If distance of any point P from the O is 'r' and $\angle XOP = \theta$, then (r, θ) are called the polar co-ordinates of a point P.



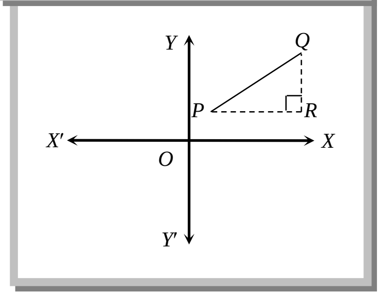
If (x, y) are the cartesian co-ordinates of a point P, then

$x = r\cos\theta$; $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

1.4 Distance Formula

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by



$$PQ = \sqrt{(PR)^2 + (QR)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : ❑ The distance of a point $M(x_0, y_0)$ from origin $O(0, 0)$

$OM = \sqrt{(x_0^2 + y_0^2)}$.

❑ If distance between two points is given then use $\pm$ sign.

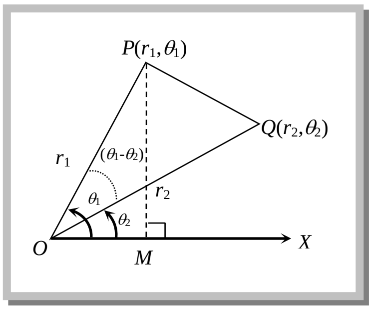
❑ When the line PQ is parallel to the y-axis, the abscissa of point P and Q will be equal *i.e*, $x_1 = x_2$;

$\therefore\ PQ = |y_2 - y_1|$

❑ When the segment PQ is parallel to the x-axis, the ordinate of the points P and Q will be equal *i.e.*, $y_1 = y_2$. Therefore $PQ = |x_2 - x_1|$

(1) **Distance between two points in polar co-ordinates :** Let O be the pole and OX be the initial line. Let P and Q be two given points whose polar co-ordinates are (r_1, θ_1) and (r_2, θ_2) respectively.

Then $OP = r_1, OQ = r_2$



$\angle POX = \theta_1$ and $\angle QOX = \theta_2$

then $\angle POQ = (\theta_1 - \theta_2)$

In ΔPOQ, from cosine rule $\cos(\theta_1 - \theta_2) = \frac{(OP)^2 + (OQ)^2 - (PQ)^2}{2OP.OQ}$

$\therefore\ (PQ)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$\therefore\ PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$

Note : ❑ Always taking θ_1 and θ_2 in radians.

Example: 1 If the point (x, y) be equidistant from the points $(a+b, b-a)$ and $(a-b, a+b)$, then **[MP PET 1983, 94]**

(a) $ax + by = 0$ (b) $ax - by = 0$ (c) $bx + ay = 0$ (d) $bx - ay = 0$

Solution: (d) Let points $P(x, y)$, $A(a+b, b-a)$, $B(a-b, a+b)$.

According to Question, $PA = PB$, *i.e.*, $PA^2 = PB^2$

$\Rightarrow (a+b-x)^2 + (b-a-y)^2 = (a-b-x)^2 + (a+b-y)^2$

$\Rightarrow (a+b)^2 + x^2 - 2x(a+b) + (b-a)^2 + y^2 - 2y(b-a) = (a-b)^2 + x^2 - 2x(a-b) + (a+b)^2 + y^2 - 2y(a+b)$

$\Rightarrow 2x(a-b-a-b) = 2y(b-a-a-b) \Rightarrow -4bx = -4ay \Rightarrow bx - ay = 0$

Example: 2 If cartesian co-ordinates of any point are $(\sqrt{3}, 1)$, then its polar co-ordinates is

(a) $(2, \pi/3)$ (b) $(\sqrt{2}, \pi/6)$ (c) $(2, \pi/6)$ (d) None of these

Solution: (c) We know that $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \sqrt{3} = r\cos\theta, \ 1 = r\sin\theta$

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$

Polar co-ordinates $= (2, \pi/6)$.

1.5 Geometrical Conditions

(1) Properties of triangles

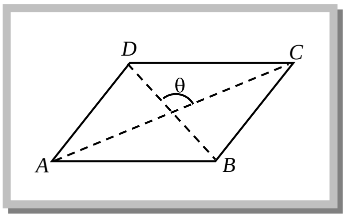
(i) In any triangle ABC, $AB + BC > AC$ and $|AB - BC| < AC$.

(ii) The ΔABC is equilateral $\Leftrightarrow AB = BC = CA$.

(iii) The ΔABC is a right angled triangle $\Leftrightarrow AB^2 = AC^2 + BC^2$ or $AC^2 = AB^2 + BC^2$ or $BC^2 = AB^2 + AC^2$.

(iv) The ΔABC is isosceles $\Leftrightarrow AB = BC$ or $BC = CA$ or $AB = AC$.

(2) Properties of quadrilaterals



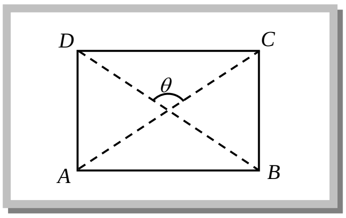
(i) The quadrilateral $ABCD$ is a parallelogram if and only if

(a) $AB = DC, AD = BC$, or (b) the middle points of BD and AC are the same,

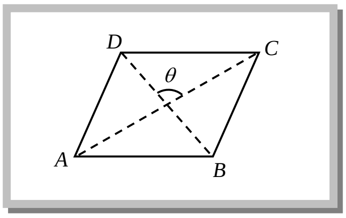
In a parallelogram diagonals AC and BD are not equal and $\theta \neq \frac{\pi}{2}$.

(ii) The quadrilateral $ABCD$ is a rectangle if and only if

(a) $AB = CD, AD = BC$ and $AC^2 = AB^2 + BC^2$ or, (b) $AB = CD, AD = BC, AC = BD$ or, (c) the middle points of AC and BD are the same and $AC=BD$. $(\theta \neq \pi/2)$

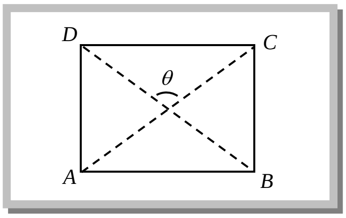


(iii) The quadrilateral $ABCD$ is a rhombus (but not a square) if and only if (a) $AB = BC = CD = DA$ and $AC \neq BD$ or, (b) the middle points of AC and BD are the same and $AB = AD$ but $AC \neq BD$. $(\theta = \pi/2)$



(iv) The quadrilateral $ABCD$ is a square if and only if

(a) $AB = BC = CD = DA$ and $AC = BD$ or (b) the middle points of AC and BD are the same and $AC = BD$, $(\theta = \pi/2)$, $AB = AD$.



Note :
- ❑ Diagonals of square, rhombus, rectangle and parallelogram always bisect each other.
- ❑ Diagonals of rhombus and square bisect each other at right angle.
- ❑ Four given points are collinear, if area of quadrilateral is zero.

Example: 3 ABC is an isosceles triangle. If the co-ordinates of the base are $B(1,3)$ and $C(-2,7)$ the co-ordinates of vertex A can be **[Orissa JEE 2002]**

(a) (1, 6) (b) $\left(-\frac{1}{2}, 5\right)$ (c) $\left(\frac{5}{6}, 6\right)$ (d) None of these

Solution: (c) Let the vertex of triangle be $A(x, y)$.

Then the vertex $A(x, y)$ is equidistant from *B* and *C* because *ABC* is an isosceles triangle, therefore

$(x-1)^2 + (y-3)^2 = (x+2)^2 + (y-7)^2 \Rightarrow 6x - 8y + 43 = 0$

Thus, any point lying on this line can be the vertex A except the mid point $\left(-\frac{1}{2}, 5\right)$ of *BC*. Hence vertex *A* is $\left(\frac{5}{6}, 6\right)$

Example: 4 The extremities of diagonal of parallelogram are the points (3, – 4) and (– 6,5) if third vertex is (– 2,1), then fourth vertex is

[Rajasthan PET 1987]

(a) (1,0) (b) (–1,0) (c) (1,1) (d) None of these

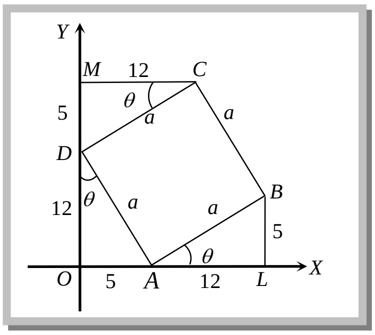
Solution: (b) Let $A(3, -4)$ and $C(-6, 5)$ be the ends of diagonal of parallelogram *ABCD*. Let $B(-2, 1)$ and *D* be (*x, y*), then mid points of diagonal *AC* and *BD* coincide. So, $\frac{x-2}{2} = \frac{-6+3}{2}$ and $\frac{y+1}{2} = \frac{5-4}{2}$

$x = -1,\ y = 0$. $\therefore$ Coordinates of *D* are (–1, 0)

Example: 5 The vertices *A* and *D* of square *ABCD* lie on positive side of *x* and *y*-axis respectively. If the vertex *C* is the point (12, 17), then the coordinate of vertex *B* are

(a) (14, 16) (b) (15, 3)

(c) (17, 5) (d) (17, 12)



Solution: (c) Let the co-ordinate of *B* be (*h, k*)

Draw *BL* and *CM* perpendicular to *x*-axis and *y*-axis.

$\therefore\ a\cos\theta = CM = OD = AL = 12$

and $a\sin\theta = DM = OA = BL = 5$

$\therefore\ k = BL = DM = OM - OD = 17 - 12 = 5$

$\therefore\ h = OL = OA + AL = 5 + 12 = 17$

Hence, Point *B* is (17, 5).

Example: 6 A triangle with vertices (4, 0); (–1, –1); (3, 5) is **[AIEEE 2002]**

(a) Isosceles and right angled (b) Isosceles but not right angled

(c) Right angled but not isosceles(d) Neither right angled nor isosceles

Solution: (a) Let A $(4,0)$; $B(-1,-1)$; $C(3,5)$ then

$AB = \sqrt{26},\ AC = \sqrt{26},\ BC = \sqrt{52}$; *i.e.* $AB = AC$

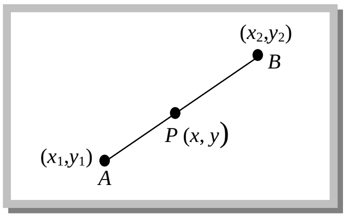
So triangle is isosceles and also $(BC)^2 = (AB)^2 + (AC)^2$. Hence ΔABC is right angled isosceles triangle.

1.6 Section Formulae

If $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$

(1) **Internal division :** If $P(x, y)$ divides the segment *AB* internally in the ratio of $m_1 : m_2$

$$\Rightarrow \frac{PA}{PB} = \frac{m_1}{m_2}$$

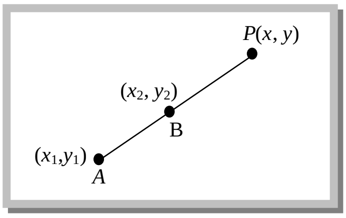


The co-ordinates of $P(x, y)$ are

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

(2) **External division :** If $P(x, y)$ divides the segment *AB* externally in the ratio of $m_1 : m_2$

$\Rightarrow \quad \frac{PA}{PB} = \frac{m_1}{m_2}$



The co-ordinates of $P(x,y)$ are $x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$ and $y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$

Note : ❑ If $P(x,y)$ divides the join of $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $\lambda : 1 (\lambda > 0)$, then $x = \frac{\lambda x_2 \pm x_1}{\lambda \pm 1}; y = \frac{\lambda y_2 \pm y_1}{\lambda \pm 1}$. Positive sign is taken for internal division and negative sign is taken for external division.

❑ The mid point of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ [Here $m_1 : m_2 :: 1:1$]

❑ For finding ratio, use ratio $\lambda : 1$. If λ is positive, then divides internally and if λ is negative, then divides externally.

❑ Straight line $ax + by + c = 0$ divides the join of points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $\left(-\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$.

If ratio is –*ve* then divides externally and if ratio is +*ve* then divides internally.

Example: 7 The co-ordinate of the point dividing internally the line joining the points (4, –2) and (8, 6) in the ratio 7: 5 will be **[AMU 1979; MP PET 1984]**

(a) (16, 18) (b) (18, 16) (c) $\left(\frac{19}{3}, \frac{8}{3}\right)$ (d) $\left(\frac{8}{3}, \frac{19}{3}\right)$

Solution: (c) Let point (*x*, *y*) divides the line internally.

Then $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{7(8)+5(4)}{12} = \frac{19}{3}$, $y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{7(6)+5(-2)}{12} = \frac{8}{3}$.

Example: 8 The line $x + y = 4$ divides the line joining the points (–1,1) and (5, 7) in the ratio **[IIT 1965, UPSEAT 1999]**

(a) 2 : 1 (b) 1 : 2 Internally (c) 1 : 2 Externally (d) None of these

Solution: (b) Required ratio $= -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) = -\left(\frac{-1+1-4}{5+7-4}\right) = \frac{4}{8} = \frac{1}{2}$ (Internally)

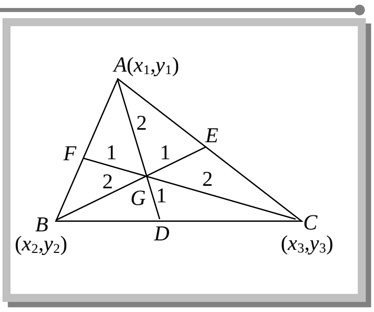
Example: 9 The line joining points (2, –3) and (–5, 6)is divided by *y*-axis in the ratio **[MP PET 1999]**

(a) 2 : 5 (b) 2 : 3 (c) 3 : 5 (d) 1 : 2

Solution: (a) Let ratio be $k : 1$ and coordinate of *y*-axis are (0, *b*). Therefore, $0 = \frac{k(-5)+1(2)}{k+1} \Rightarrow k = \frac{2}{5}$

1.7 Some points of a Triangle

(1) **Centroid of a triangle :** The centroid of a triangle is the point of intersection of its medians. The centroid divides the medians in the ratio $2:1$ (Vertex : base)



If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a triangle. If G be the centroid upon one of the median (say) AD, then $AG : GD = 2 : 1$

$\Rightarrow$ Co-ordinate of G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Example: 10 The centroid of a triangle is (2,7)and two of its vertices are (4, 8) and (–2, 6) the third vertex is **[Kerala (Engg.) 2002]**

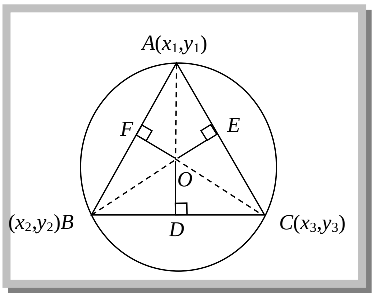
(a) (0, 0) (b) (4, 7) (c) (7, 4) (d) (7, 7)

Solution: (b) Let the third vertex (x, y)

$$2 = \frac{x+4-2}{3}, 7 = \frac{y+8+6}{3}, \text{ i.e. } x = 4, \; y = 7$$

Hence third vertex is (4, 7).

(2) **Circumcentre :** The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the circum-radius of the triangle.



Let vertices *A*, *B*, *C* of the triangle *ABC* be $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) and let circumcentre be $O(x, y)$ and then (x, y) can be found by solving

$$(OA)^2 = (OB)^2 = (OC)^2$$

i.e., $(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2 = (x - x_3)^2 + (y - y_3)^2$

Note : ❑ If a triangle is right angle, then its circumcentre is the mid point of hypotenuse.

❑ If angles of triangle *i.e.*, *A*, *B*, *C* and vertices of triangle $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then circumcentre of the triangle *ABC* is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Example: 11 If the vertices of a triangle be (2, 1); (5, 2) and (3,4) then its circumcentre is **[IIT 1964]**

(a) $\left(\frac{13}{2}, \frac{9}{2}\right)$ (b) $\left(\frac{13}{4}, \frac{9}{4}\right)$ (c) $\left(\frac{9}{4}, \frac{13}{4}\right)$ (d) None of these

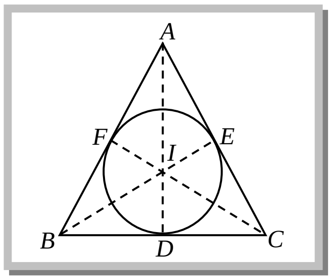
Solution: (b) Let circumcentre be $O(x, y)$ and given points are $A(2,1); B(5,2); C(3,4)$ and $OA^2 = OB^2 = OC^2$

$\therefore (x-2)^2 + (y-1)^2 = (x-5)^2 + (y-2)^2$ .....(i)

and $(x-2)^2 + (y-1)^2 = (x-3)^2 + (y-4)^2$ .....(ii)

On solving (i) and (ii), we get $x = \frac{13}{4}, y = \frac{9}{4}$

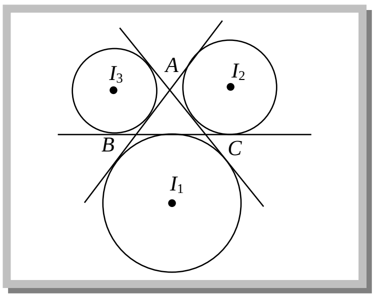
(3) **Incentre :** The incentre of a triangle is the point of intersection of internal bisector of the angles. Also it is a centre of a circle touching all the sides of a triangle.



Co-ordinates of incentre $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Where *a*, *b*, *c* are the sides of triangle *ABC*.

(4) **Excircle :** A circle touches one side outside the triangle and other two extended sides then circle is known as excircle. Let ABC be a triangle then there are three excircles with three excentres. Let I_1, I_2, I_3 opposite to vertices *A*,*B* and *C* respectively. If vertices of triangle are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ then



$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right), \; I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Note : ❑ Angle bisector divides the opposite sides in the ratio of remaining sides *e.g.* $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

- Incentre divides the angle bisectors in the ratio $(b+c):a, (c+a):b$ and $(a+b):c$
- **Excentre :** Point of intersection of one internal angle bisector and other two external angle bisector is called as excentre. There are three excentres in a triangle. Co-ordinate of each can be obtained by changing the sign of a,b,c respectively in the formula of in-centre.

Example: 12 The incentre of the triangle with vertices $(1,\sqrt{3}),(0,0)$ and $(2,0)$ is **[IIT Screening 2000]**

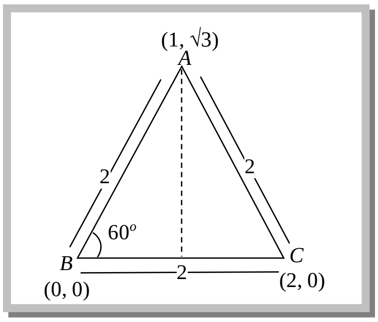
(a) $\left(1,\frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3},\frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3},\frac{\sqrt{3}}{2}\right)$ (d) $\left(1,\frac{1}{\sqrt{3}}\right)$

Solution: (d) $\because$ Here $AB = BC = CA$

$\therefore$ The triangle is equilateral .

So, the incentre is the same as the centroid.

$\therefore$ Incentre $=\left(\frac{1+0+2}{3},\frac{\sqrt{3}+0+0}{3}\right)=\left(1,\frac{1}{\sqrt{3}}\right)$.

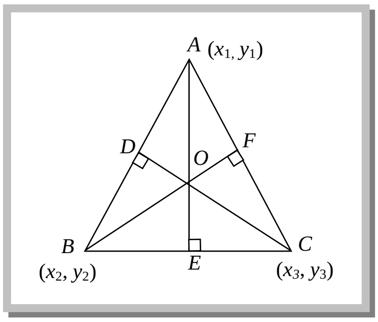


(5) **Orthocentre :** It is the point of intersection of perpendiculars drawn from vertices on opposite sides (called altitudes) of a triangle and can be obtained by solving the equation of any two altitudes.

Here O is the orthocentre since $AE \perp BC$, $BF \perp AC$ and $CD \perp AB$

then $OE \perp BC, OF \perp AC, OD \perp AB$

Solving any two we can get coordinate of O.



Note : If a triangle is right angled triangle, then orthocentre is the point where right angle is formed.

- If the triangle is equilateral then centroid, incentre, orthocentre, circum-centre coincides.
- Orthocentre, centroid and circum-centre are always collinear and centroid divides the line joining orthocentre and circum-centre in the ratio 2 : 1
- In an isosceles triangle centroid, orthocentre, incentre, circum-centre lie on the same line.

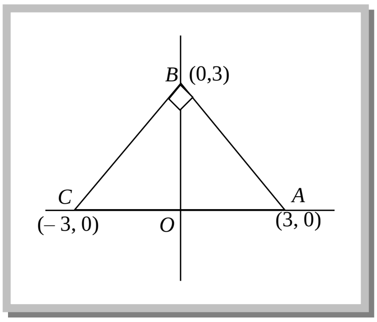
Example: 13 The vertices of triangle are (0, 3) (– 3, 0) and (3, 0). The co-ordinate of its orthocentre are **[AMU 1991; DCE 1994]**

(a) (0, – 2) (b) (0, 2) (c) (0, 3) (d) (0, –3)

Solution: (c) Here $AB \perp BC$.

In a right angled triangle, orthocentre is the point where right angle is formed.

$\therefore$ Orthocentre is (0, 3)



Example: 14 If the centroid and circumcentre of triangle are (3, 3); (6, 2), then the orthocentre is **[DCE 2000]**

(a) (9, 5) (b) (3, –1) (c) (– 3, 1) (d) (– 3, 5)

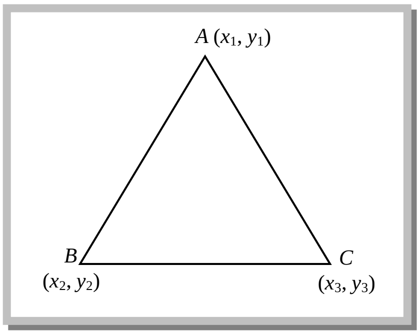
Solution: (d) Let orthocentre be (α, β). We know that centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

$$\therefore 3 = \frac{2(6)+1(\alpha)}{2+1} \Rightarrow \alpha = -3,\ 3 = \frac{2(2)+1(\beta)}{2+1} \Rightarrow \beta = 5$$

Hence orthocentre is (–3, 5).

1.8 Area of some Geometrical figures

(1) **Area of a triangle :** The area of a triangle ABC with vertices $A(x_1, y_1); B(x_2, y_2)$ and $C(x_3, y_3)$. The area of triangle ABC is denoted by 'Δ' and is given as



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

In equilateral triangle

(i) Having sides a, area is $\frac{\sqrt{3}}{4}a^2$.

(ii) Having length of perpendicular as 'p' area is $\frac{(p^2)}{\sqrt{3}}$.

Note : ❑ If a triangle has polar co-ordinates $(r_1, \theta_1), (r_2, \theta_2)$ and (r_3, θ_3) then its area

$$\Delta = \frac{1}{2}[r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3)]$$

❑ If area is a rational number. Then the triangle cannot be equilateral.

(2) **Collinear points :** Three points $A(x_1, y_1); B(x_2, y_2); C(x_3, y_3)$ are collinear. If area of triangle is zero,

i.e., (i) $\Delta = 0 \Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(ii) $AB + BC = AC$ or $AC + BC = AB$ or $AC + AB = BC$

(3) **Area of a quadrilateral :** If $(x_1, y_1); (x_2, y_2); (x_3, y_3)$ and (x_4, y_4) are vertices of a quadrilateral, then its

$$\text{Area} = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + (x_4 y_1 - x_1 y_4)]$$

Note : ❑ If two opposite vertex of rectangle are (x_1, y_1) and (x_2, y_2), then its area is $|(y_2 - y_1)(x_2 - x_1)|$.

❑ It two opposite vertex of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area is

$$= \frac{1}{2}AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

(4) **Area of polygon :** The area of polygon whose vertices are $(x_1, y_1); (x_2, y_2); (x_3, y_3); \ldots (x_n, y_n)$ is

$$=\frac{1}{2}|\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+....+(x_ny_1-x_1y_n)\}|$$

Or **Stair method :** Repeat first co-ordinates one time in last for down arrow use positive sign and for up arrow use negative sign.

$$\therefore \quad \text{Area of polygon} = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ : & : \\ : & : \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right| = \frac{1}{2}|\{(x_1y_2+x_2y_3+....+x_ny_1)-(y_1x_2+y_2x_3+....+y_nx_1)\}|$$

Example: 15 The area of the triangle formed by the points $(a, b+c), (b, c+a), (c, a+b)$ is **[IIT 1963; EAMCET 1982; Rajasthan PET 2003]**

(a) abc (b) $a^2+b^2+c^2$ (c) $ab+bc+ca$ (d) 0

Solution: (d) Area $=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$, (Applying $C_2 \to C_1+C_2$) $=\frac{a+b+c}{2}\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}=0$

Example: 16 Three points are $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $P(x, y)$ is a point, then the ratio of area of ΔPBC and ΔABC is **[IIT 1983]**

(a) $\left|\frac{x+y-2}{7}\right|$ (b) $\left|\frac{x-y+2}{2}\right|$ (c) $\left|\frac{x-y-2}{7}\right|$ (d) None of these

Solution: (a) $\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC}=\frac{\frac{1}{2}[-3(-2-y)+4(y-5)+x(5+2)]}{\frac{1}{2}[6(5+2)-3(-2-3)+4(3-5)]}=\left|\frac{7x+7y-14}{49}\right|=\left|\frac{x+y-2}{7}\right|$

Example: 17 If the points $(2K, K)(K, 2K)$ and (K, K) with $K>0$ enclose triangle of area 18 square units then the centroid of triangle is equal to

(a) (8, 8) (b) (4, 4) (c) $(-4, -4)$ (d) $(4\sqrt{2}, 4\sqrt{2})$

Solution: (a) $\Delta=\frac{1}{2}\begin{vmatrix} 2K & K & 1 \\ K & 2K & 1 \\ K & K & 1 \end{vmatrix}=18 \Rightarrow \frac{K^2}{2}=18 \Rightarrow K=\pm 6$. Consider $K=+6$ because $K>0$, then the points (12, 6) (6,12) and (6,6).

Hence, centroid $=\left(\frac{12+6+6}{3}, \frac{6+12+6}{3}\right)=(8, 8)$

Example: 18 If the points $(x+1, 2)$; $(1, x+2)$; $\left(\frac{1}{x+1}, \frac{2}{x+1}\right)$ are collinear, then x is **[Rajasthan PET 2002]**

(a) 4 (b) 0 (c) -4 (d) None of these

Solution: (b,c) Let $A=(x+1, 2)$; $B=(1, x+2)$; $C=\left(\frac{1}{x+1}, \frac{2}{x+1}\right)$. A, B, C are collinear, if area of $\Delta ABC=0$

$$\Rightarrow \begin{vmatrix} x+1 & 2 & 1 \\ 1 & x+2 & 1 \\ \frac{1}{x+1} & \frac{2}{x+1} & 1 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x & -x & 0 \\ 1 & x+2 & 1 \\ \frac{1}{x+1} & \frac{2}{x+1} & 1 \end{vmatrix}=0 \qquad (R_1 \to R_1-R_2)$$

$$\Rightarrow \begin{vmatrix} x & 0 & 0 \\ 1 & x+3 & 1 \\ \frac{1}{x+1} & \frac{3}{x+1} & 1 \end{vmatrix}=0 \quad (C_2 \to C_1+C_2) \Rightarrow x^2(x+4)=0 \Rightarrow x=0, -4$$

Example: 19 The points (1, 1); $(0, \sec^2\theta)$; $(\text{cosec}^2\theta, 0)$ are collinear for **[Roorkee 1963]**

(a) $\theta=n\pi/2$ (b) $\theta \neq n\pi/2$ (c) $\theta=n\pi$ (d) None of these

Solution: (b) The given points are collinear, if Area of $\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sec^2\theta & 1 \\ \text{cosec}^2\theta & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1(\sec^2\theta) + 1(\text{cosec}^2\theta) + 1(-\text{cosec}^2\theta.\sec^2\theta) = 0$

$\Rightarrow \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} - \frac{1}{\sin^2\theta.\cos^2\theta} = 0 \Rightarrow \frac{1}{\sin^2\theta.\cos^2\theta} - \frac{1}{\sin^2\theta.\cos^2\theta} = 0 \Rightarrow 0 = 0$

Therefore the points are collinear for all value of θ, except only $\theta = \frac{n\pi}{2}$ because at $\theta = \frac{n\pi}{2}$, $\sec^2\theta = \infty$ (Not defined).

Example: 20 The points (0, 8/3) (1, 3) and (82, 30) are the vertices of **[IIT 1983; Rajasthan PET 1988]**

(a) An equilateral triangle
(b) An isosceles triangle
(c) A right angled triangle
(d) None of these

Solution: (d) Here $A = (0, 8/3)$, $B = (1, 3)$ and $C = (82, 30)$

$AB = \sqrt{1 + 1/9} = \sqrt{10/9}$, $BC = \sqrt{(81)^2 + (27)^2} = 27\sqrt{10} = 81\sqrt{\frac{10}{9}}$, $AC = \sqrt{(82)^2 + (30 - 8/3)^2} = 82\sqrt{\frac{10}{9}}$

Since $AB + BC = (1 + 81)\sqrt{\frac{10}{9}} = 82\sqrt{\frac{10}{9}} = AC$. $\therefore$ Points A, B, C are collinear.

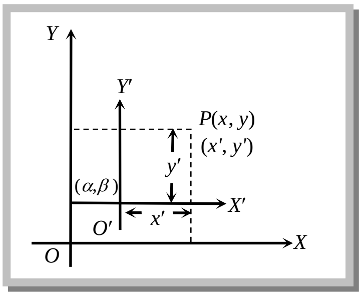
1.9 Transformation of Axes

(1) **Shifting of origin without rotation of axes :** Let $P \equiv (x, y)$ with respect to axes OX and OY.

Let $O' \equiv (\alpha, \beta)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes $O'X'$ and $O'Y'$, where OX and $O'X'$ are parallel and OY and $O'Y'$ are parallel.

Then $x = x' + \alpha, y = y' + \beta$ or $x' = x - \alpha, y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.



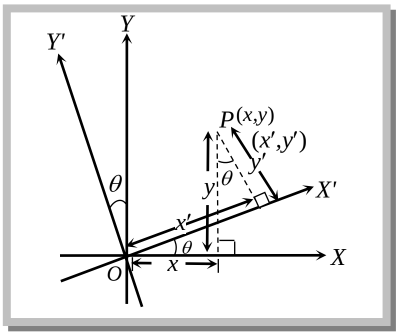
(2) **Rotation of axes without changing the origin :** Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$

then $x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

and $x' = x\cos\theta + y\sin\theta$

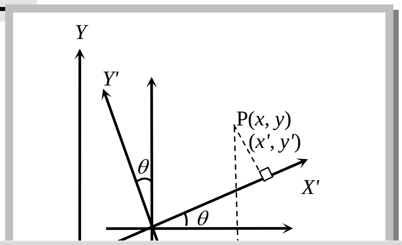
$y' = -x\sin\theta + y\cos\theta$



The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

(3) **Change of origin and rotation of axes :** If origin is changed to $O'(\alpha, \beta)$ and axes are rotated about the new origin O' by an angle θ in the



anticlock-wise sense such that the new co-ordinates of $P(x,y)$ become (x',y') then the equations of transformation will be $x=\alpha+x'\cos\theta-y'\sin\theta$ and $y=\beta+x'\sin\theta+y'\cos\theta$

(4) **Reflection (Image of a point) :** Let (x,y) be any point, then its image with respect to

(i) x axis $\Rightarrow (x,-y)$ (ii) y-axis $\Rightarrow (-x,y)$ (iii) origin $\Rightarrow (-x,-y)$ (iv) line $y=x \Rightarrow (y,x)$

Example: 21 The point (2,3) undergoes the following three transformation successively,

(i) Reflection about the line $y=x$.

(ii) Transformation through a distance 2 units along the positive direction of y-axis.

(iii) Rotation through an angle of 45° about the origin in the anticlockwise direction.

The final coordinates of points are **[Roorkee 2000]**

(a) $\left(\frac{1}{\sqrt{2}},\frac{7}{\sqrt{2}}\right)$ (b) $\left(\frac{-1}{\sqrt{2}},\frac{7}{\sqrt{2}}\right)$ (c) $\left(\frac{1}{\sqrt{2}},\frac{-7}{\sqrt{2}}\right)$ (d) None of these

Solution: (b) (i) The new position after reflection is (3,2)

(ii) After transformation, it is (3, 2+ 2), *i.e*, (3, 4)

(iii) Rotation makes it $(3\cos 45^o-4\sin 45^o, 3\sin 45^o+4\cos 45^o)$, *i.e.* $\left(\frac{-1}{\sqrt{2}},\frac{7}{\sqrt{2}}\right)$

Example: 22 Reflecting the point (2, –1) about y-axis, coordinate axes are rotated at 45^o angle in negative direction without shifting the origin. The new coordinates of the point are

(a) $\left(\frac{-1}{\sqrt{2}},\frac{-3}{\sqrt{2}}\right)$ (b) $\left(\frac{-3}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (c) $\left(\frac{1}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ (d) None of these

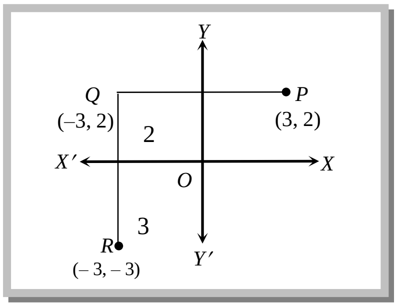
Solution: (a) The new position after reflection is (–2, –1)

Rotation makes it $[(-2)\cos(-45^o)+(-1)\sin(-45^o), -(-2)\sin(-45^o)+(-1)\cos(-45^o)]$, *i.e.*, $\left[\frac{-1}{\sqrt{2}},\frac{-3}{\sqrt{2}}\right]$

Example: 23 The point (3, 2) is reflected in the y-axis and then moved a distance 5 units towards the negative side of y-axis. The co-ordinate of the point thus obtained are **[DCE 1997]**

(a) (3, –3) (b) (–3, 3) (c) (3, 3) (d) (–3, –3)

Solution: (d) Reflection in the y-axis of the point (3,2) is (–3, 2) when it moves towards the negative side of y- axis through 5 units, then the new position is (–3, 2– 5) =(– 3, – 3)



1.10 Locus

Locus : The curve described by a point which moves under given condition or conditions is called its locus.

Equation to the locus of a point : The equation to the locus of a point is the relation, which is satisfied by the coordinates of every point on the locus of the point.

Algorithm to find the locus of a point

Step I : Assume the coordinates of the point say (h, k) whose locus is to be found.

Step II : Write the given condition in mathematical form involving h , k.

Step III : Eliminate the variable (s), if any.

Step IV : Replace h by x and k by y in the result obtained in step III. The equation so obtained is the locus of the point which moves under some stated condition (s)

Note : ❑ Locus of a point P which is equidistant from the two point A and B is a straight line and is a perpendicular bisector of line AB.

❑ In above case if $PA = kPB$ where $k \neq 1$, then the locus of P is a circle.

❑ Locus of P if A and B is fixed.

(a) Circle, if $\angle APB$ = constant (b) Circle with diameter AB, if $\angle APB = \frac{\pi}{2}$

(c) Ellipse, if $PA + PB$ = constant (d) Hyperbola, if $PA - PB$ = constant

Example: 24 Let A (2, – 3) and B(– 2, 1) be vertices of triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line **[AIEEE 2004]**

(a) $3x - 2y = 3$ (b) $2x - 3y = 7$ (c) $3x + 2y = 5$ (d) $2x + 3y = 9$

Solution: (d) Let third vertex C be (α, β)

$\therefore$ Centroid $= \left(\frac{2-2+\alpha}{3}, \frac{-3+1+\beta}{3}\right)$, *i.e.* $\left(\frac{\alpha}{3}, \frac{\beta-2}{3}\right)$

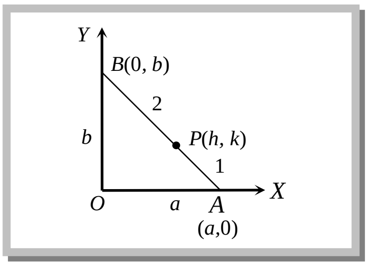
According to question, $2\left(\frac{\alpha}{3}\right) + 3\left(\frac{\beta-2}{3}\right) = 1 \Rightarrow 2\alpha + 3\beta - 6 = 3 \Rightarrow 2\alpha + 3\beta = 9$

Hence, locus of vertex C is $2x + 3y = 9$.

Example: 25 The ends of a rod of length l move on two mutually perpendicular lines. The locus of the point on the rod which divides it in the ratio 1 : 2 is **[IIT 1987; Rajasthan PET 1997]**

(a) $36x^2 + 9y^2 = 4l^2$ (b) $36x^2 + 9y^2 = l^2$ (c) $9x^2 + 36y^2 = 4l^2$ (d) None of these

Solution: (c) $AP : PB = 1 : 2$, then $h = \frac{1 \times 0 + 2 \times a}{1+2} = \frac{2a}{3}$ or $a = \frac{3h}{2}$, Similarly $b = 3k$



Now we have $OA^2 + OB^2 = AB^2 \Rightarrow \left(\frac{3h}{2}\right)^2 + (3k)^2 = l^2$

Hence locus of P (h, k) is given by $9x^2 + 36y^2 = 4l^2$

Example: 26 If A and B are two fixed points and P is a variable point such that $PA + PB = 4$, then the locus of P is a/an **[IIT 1989; UPSEAT 2001]**

(a) Parabola (b) Ellipse (c) Hyperbola (d) None of these

Solution: (b) We know that, $PA + PB$ = constant. Then locus of P is an ellipse.

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